Stage 11
Prompt Sheet

11/1 Simplify surds

$\sqrt{25}$ is NOT a surd because it is exactly 5

$\sqrt{3}$ is a surd because the answer is not exact

A surd is an irrational number

- **To simplify surds look for square number factors**

$\sqrt{75} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$

11/2 Manipulate expressions in surds

Add & subtract

$m\sqrt{a} \pm n\sqrt{a} = (m\pm n)\sqrt{a}$

Example 1 $2\sqrt{5} + 3\sqrt{5} = (2+3)\sqrt{5} = 5\sqrt{5}$

Multiply & divide

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Example 2 $\sqrt{3} \times \sqrt{15} = \sqrt{45} = 3\sqrt{5}$

Example 3 $(4 + \sqrt{3})(2 - \sqrt{3})$

$= 8 - 4\sqrt{3} + 2\sqrt{3} - 3$

$= 5 - 2\sqrt{3}$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example 4 $\sqrt{\frac{72}{20}} = \frac{\sqrt{72}}{\sqrt{20}} = \frac{6\sqrt{2}}{2\sqrt{5}} = \frac{3\sqrt{2}}{\sqrt{5}}$

11/2 Rationalise surd denominators

To remove a surd from the denominator multiply the numerator & the denominator by that surd

Example 5

$\frac{6}{\sqrt{12}}$ (Multiply both top & bottom by √12)

$= \frac{6}{\sqrt{12}} \times \frac{\sqrt{12}}{\sqrt{12}}$

$= \frac{6\sqrt{12}}{12}$ (Cancel by 6)

$= \frac{\sqrt{12}}{2}$

$= \frac{2\sqrt{3}}{2}$

$= \sqrt{3}$

11/3 Calculate with upper & lower bounds

- **If 'a' is rounded to nearest 'x'**

Upper bound = $a + \frac{1}{2}x$

Lower bound = $a - \frac{1}{2}x$

Example: if 1.8 is rounded to 1dp

Upper bound = $1.8 + \frac{1}{2}(0.1) = 1.85$

Lower bound = $1.8 - \frac{1}{2}(0.1) = 1.75$

- **Calculating using bounds**

Adding bounds

Maximum = Upper + upper

Minimum = Lower + lower

Subtracting bounds

Maximum = Upper - lower

Minimum = Lower – upper

Multiplying

Maximum = Upper x upper

Minimum = Lower x lower

Dividing

Maximum = Upper ÷ lower

Minimum = Lower ÷ upper

11/4 Algebraic fractions

- ***Adding & subtracting algebraic fractions***

Example 1

$\frac{x+3}{4} + \frac{x-5}{3}$ (common denominator is 12)

$= \frac{3(x+3)}{12} + \frac{4(x-5)}{12}$

$= \frac{3x + 9 + 4x - 20}{12}$

$= \frac{7x - 11}{12}$

Example 2

$\frac{5}{(x+1)} - \frac{3}{(x+2)}$ (common denominator is (x+1)(x+2)

$= \frac{5(x+2) - 3(x+1)}{(x+1)(x+2)}$

$= \frac{5x + 10 - 3x - 3}{(x+1)(x+2)}$

$= \frac{2x + 7}{(x+1)(x+2)}$

- **Simplifying algebraic fractions**

Example:

$$\frac{2x^2 + 3x + 1}{x^2 - 3x - 4} \quad \text{(factorise)}$$

$$= \frac{(2x + 1)\cancel{(x + 1)}}{(x - 4)\cancel{(x + 1)}} \quad \text{(cancel)}$$

$$= \frac{(2x + 1)}{(x - 4)}$$

11/5 Solve equations with fractions

$\frac{x}{2x - 3} + \frac{4}{x + 1} = 1$ Common denominator (2x-3)(x+1)

$\frac{x(x+1) + 4(2x-3)}{(2x-3)(x+1)} = 1$

$\frac{x^2 + x + 8x - 12}{(2x-3)(x+1)} = 1$

$x^2 + 9x - 12 = 1(2x-3)(x+1)$

$x^2 + 9x - 12 = 2x^2 - x - 3$ ($-x^2$ from both sides)

$9x - 12 = x^2 - x - 3$ ($-9x$ from each side)

$-12 = x^2 - 10x - 3$ ($+12$ to each side)

$0 = x^2 - 10x + 9$ (factorise)

$(x + 9)(x + 1) = 0$

$x = -9$ or $x = -1$

11/6 Solve quadratic equation by factoring

- **Put equation in form $ax^2 + bx + c = 0$**

$2x^2 = 3x + 5 \equiv 2x^2 - 3x - 5 = 0$

- **Factorise the left hand side**

$(2x - 5)(x + 1) = 0$

- **Equate each factor to zero**

$2x - 5 = 0$ or $x + 1 = 0$

$x = 2.5$ or $x = -1$

11/7 Interpret expressions as functions

A function is a rule that takes numbers as inputs and assigns to each input exactly one number as output. The output is a function of the input.

- **Simple expressions as functions**

Example: $y = 3x + 5$

$f(x) = 3x + 5$ (Replace y with 'f of x')

$f(4) = 3(4) + 5 = 17$

This is the input into the function

This is the output of the function

- **Inverse function**

This is the reverse process that takes you back to the original values

We write the inverse of $f(x)$ as $f^{-1}(x)$

Example: if $f(x) = 3x + 5$

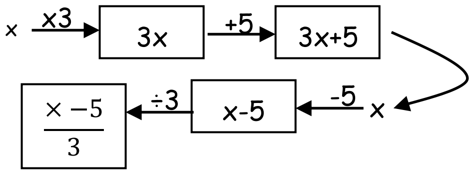
We say $y = 3x + 5$

$3x + 5 = y$

$x = \frac{y-5}{3}$ Rearrange in terms of x

$f^{-1}(x) = \frac{x-5}{3}$ Change y back to x

- **Inverse function using a flow diagram**



$f^{-1}(x) = \frac{x-5}{3}$

- **Composite function**

Applying one function to the results of another

Example 1: To combine these two functions

$f(x) = 2x$ and $g(x) = 3x - 1$

gf(x) means $g(2x) = 3(2x) - 1$

$= 6x - 1$

Replace x in the function g(x) with 2x

fg(x) means $f(3x - 1) = 2(3x - 1)$

$= 6x - 2$

Replace x in the function f(x) with 3x-1

Example 2: To evaluate the composition of functions

If $f(x) = x - 10$ and $g(x) = 2x + 3$, work out $fg(3)$

Find $g(3) = 2(3)+3 = 9$

Then $f(9) = 9-10 = -1$

11/8 Deduce turning point of quadratic functions by completing the square

- **To complete the square**

$x^2 + 4x + 4 = (x+2)^2$ – (perfect square)

$x^2 + 4x + 3 = (x+2)^2 – 1$ (completed square form)

- **Rules to complete the square**

Example 1: x^2+4x+3

$(x+2)$	x in bracket with ½ of +4
$(x+2)^2$	put squared sign on bracket
$(x+2)^2 - 4$	subt the square of the new end number
$(x+2)^2 -4+3$	add /subtract the original end number
$(x+2)^2 -1$	simplify

Example 2: $2x^2 + 6x + 5$ divide all terms by 2

$2(x^2 + 3x + 2.5)$

$2((x + 1.5)^2 -2.25) + 5$

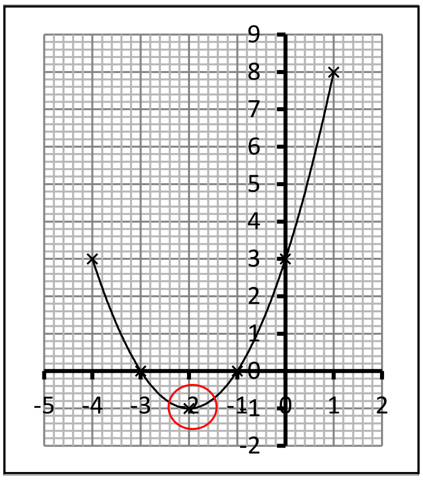
$2(x + 1.5)^2 - 4.5 + 5$

$2(x+1.5)^2 + 0.5$

- **Deduce turning point of quadratic**

Example: $y = x^2+4x+3$

$x^2 + 4x + 3 = (x+2)^2–1$	complete the square
Turning point is (-2,-1)	identify the coordinates



11/9 Solve quadratic equation by completing the square

- **Make the coefficient of x^2 a square**

  $2x^2 + 10x + 5 = 0$ (mult by 2)

  $4x^2 + 20x + 10 = 0$

- **Add a number to both sides to make a perfect square**

  $4x^2 + 20x + 10 = 0$ (Add 15)

  $4x^2 + 20x +25 = 15$

  $(2x + 5)^2 = 15$

- **Square root both sides**

  $2x + 5 = \pm\sqrt{15}$ (-5 from both sides)

  $x = \dfrac{-5 + \sqrt{15}}{2}$ OR $\dfrac{-5 - \sqrt{15}}{2}$

  $x = -0.56$ OR -4.44 (2dp)

11/10 Solve quadratic equations by formula

$$ax^2 + bx + c = 0$$

Formula (**to learn**): $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example

To solve: $3x^2 + 4x - 2 = 0$

a = 3
b = 4
c = -2

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-4 \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}$

$= \dfrac{-4 \pm \sqrt{16+24}}{6}$

$= \dfrac{-4 \pm \sqrt{40}}{6}$

$x = \dfrac{-4 + \sqrt{40}}{6}$ OR $\dfrac{-4 - \sqrt{40}}{6}$

These are **EXACT** values

x = 0.39(2dp) OR -1.72 (2dp)

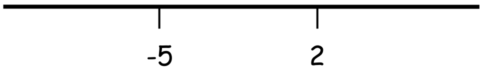
11/11 Solve quadratic inequalities

Example: $x^2 + 3x > 10$

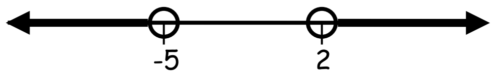
$x^2 + 3x - 10 = 0$	Replace inequality symbol with =
$(x - 2)(x + 5) = 0$	Factorise
x = 2 and x = -5	Solve

Prepare the number line

Test x=-6 in inequality $(-6)^2+3(-6)$ =18 18>10 **TRUE**	Test x=0 in inequality $(0)^2+3(0)$ =0 0>10 **FALSE**	Test x=3 in inequality $(3)^2+3(3)$ =18 18>10 **TRUE**

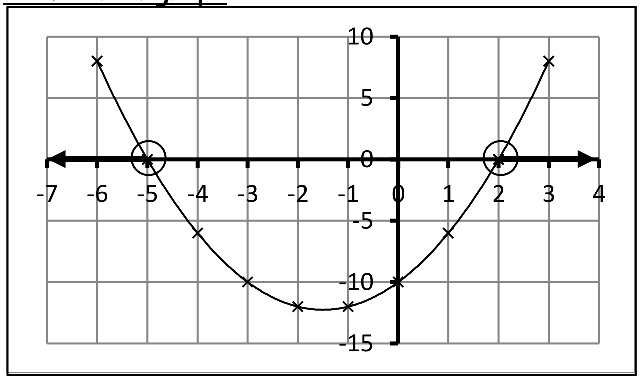


Solution set on number line



Set notation: x<-5 and x>2

Solution on graph



11/12 Rearrange more complex formulae
(inc where subject appears twice)

- Collect all the terms with the new subject
- Factorise to isolate the new subject

Example: to make 'b' the new subject

$a = \frac{2 - 7b}{b - 5}$ (multiply both sides by (b – 5))

$a(b - 5) = 2 - 7b$ (Expand the bracket)

$ab - 5a = 2 - 7b$ (Collect terms in new subject)

$7b + ab - 5a = 2$ (+5a to both sides)

$7b + ab = 2 + 5a$ (factorise to isolate 'b')

$\frac{b(7 + a)}{(7 + a)} = \frac{2 + 5a}{(7 + a)}$ (÷(7 + a) both sides)

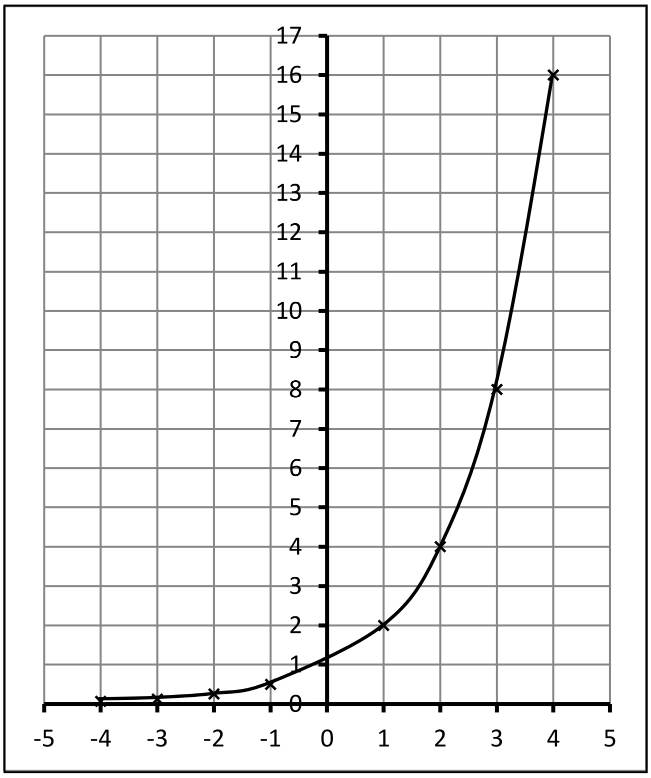
$b = \frac{2 + 5a}{(7 + a)}$

11/13 Exponential graphs

The graph of the exponential function is:

$$y = k^x$$

Example: $y = 2^x$



It has no maximum or minimum point

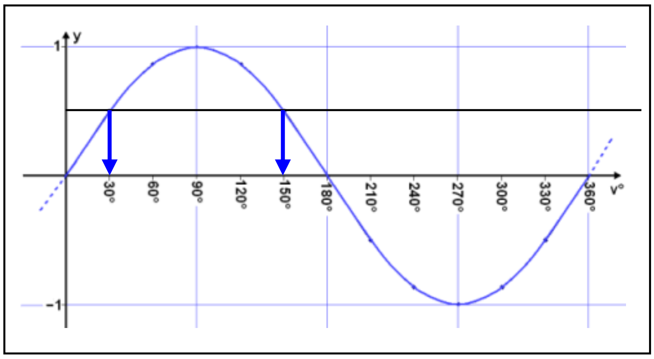
It crosses the y-axis at (0,1)

It never crosses the x-axis

11/14 Graphs of trigonometric functions

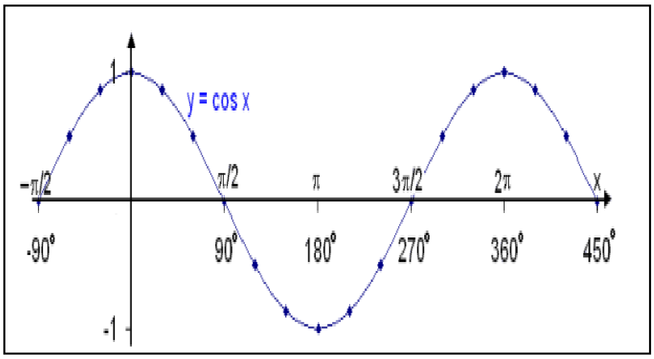
LEARN THE SHAPES OF THE GRAPHS

Graph of y=sin x



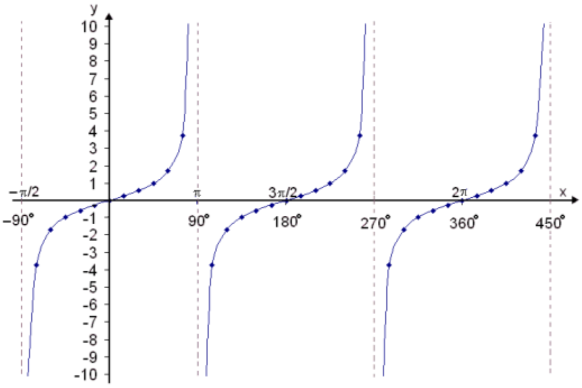
$-1 \leq \sin x \leq 1$

Graph y = cos x



$-1 \leq \cos x \leq 1$

Graph y = tan x



tan x is undefined at 90^0, 270^0 ....

Solutions to trigonometric equations can be found on the calculator and by using the symmetry of these graphs

Example:

If sin x = 0.5

$x = 30^0, 150^0$, *(See the solutions on sin graph above or from calculator)*

11/15 Transformation of functions

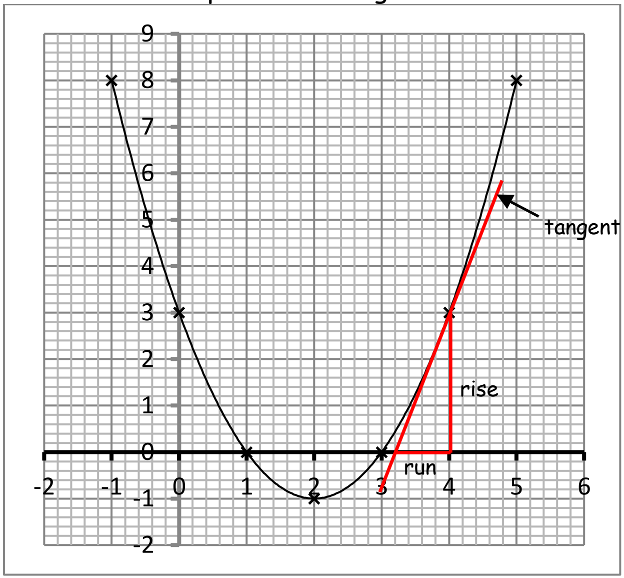
For any graph y = f(x) **LEARN** the transformations

y=f(x) ± a	Translation $\binom{0}{\pm a}$ moves up(+)/down(-)
y=f(x± a)	Translation $\binom{\pm a}{0}$ moves right(-)/left(+)
y=-f(x)	Reflection in the x-axis (horizontally)
y=f(-x)	Reflection in the y-axis (vertically)

11/16 Gradient of a curve

Example: To find gradient at point x=4

- Draw tangent at x=4 to the curve
- Pick 2 points on the tangent (x_1,y_1) & (x_2,y_2)
- Work out rise & run or use $\frac{y_2 - y_1}{x_2 - x_1}$
- Check if positive or negative



Rise = 4

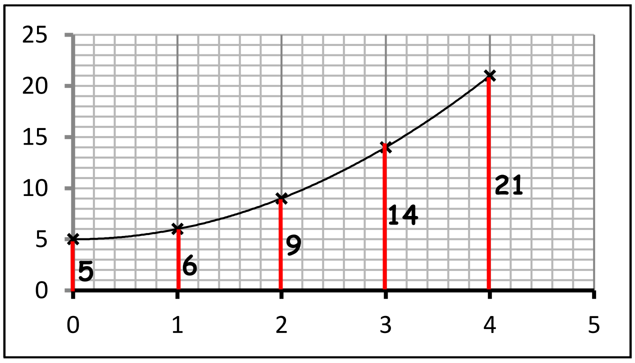
Run = 0.8

Gradient = 4÷0.8 = 40÷8 = 5

Its slope is positive

11/16 Area under a curve

- Split into trapeziums
- Find the sum of their areas



Area = $\frac{1}{2}$ x 1 x(5 +2(6 + 9 + 14) + 21)

= $\frac{1}{2}$ x 1 x (5 + 58 +21) = 42 units^2

The curve is concave, so it will be a slight over-estimate

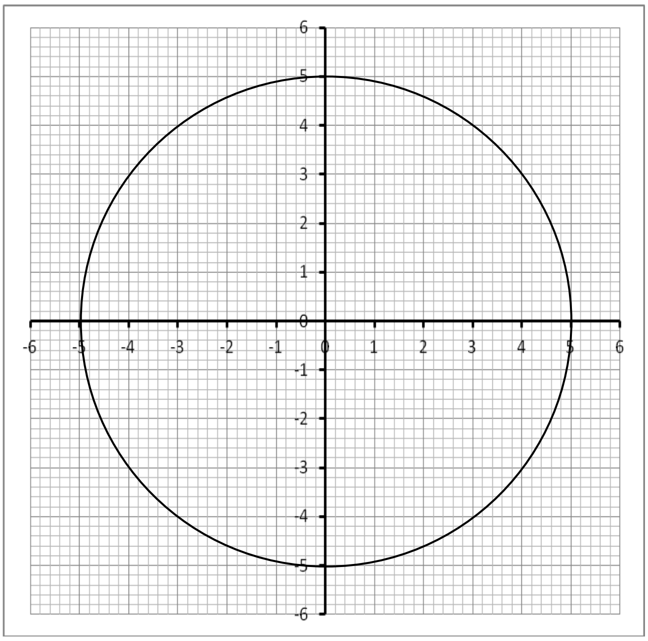
Convex curves give an over-estimate

11/17 Graph of the circle

The graph of a circle is of the form:

$$x^2 + y^2 = r^2$$

where r is the radius and the centre is (0,0)



This a circle of radius 5 and a centre (0,0)

The graph of this circle is

$$x^2 + y^2 = 5^2$$
$$x^2 + y^2 = 25$$

11/17 Equation of tangent to circle

Equation of tangent: $y - y_1 = m(x - x_1)$

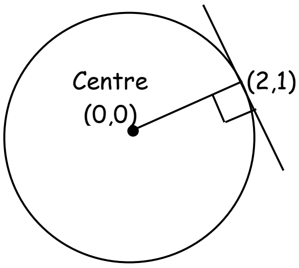
m= gradient of tangent at the point

It is perpendicular to the radius

so $m_{radius} \times m_{tangent} = -1$

(x_1, y_1) = point on circle where tangent meets

Example



Gradient of radius = $\frac{1}{2}$

Gradient of tangent = -2 ($m_{radius} \times m_{tangent} = -1$)

$y - y_1 = m(x - x_1)$

$y - 1 = -2(x - 2)$

$y - 1 = -2x + 4$

$y = -2x + 5$ **(equation of tangent)**

11/19 Solve simultaneous equations~one linear, one quadratic algebraically

- Rewrite the linear with one letter in terms of the other
- Substitute the linear into the quadratic
- Solve the quadratic by factorising

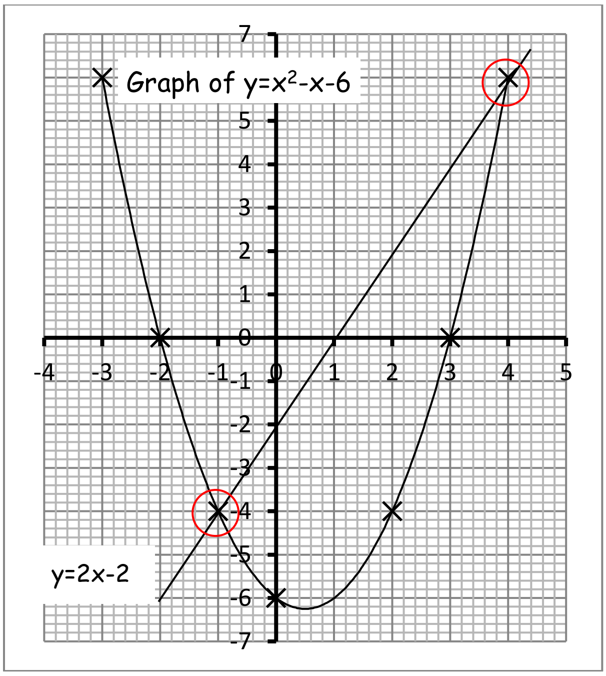
Example: To solve $y=2x-2$ and $y=x^2-x-6$

Substitute $y=2x-2$ into $y=x^2-x-6$
$2x-2 = x^2-x-6$
$x^2 -3x - 4 = 0$ (factorise)
$(x - 4)(x + 1) = 0$
$x = 4$ or $x = -1$

when $x= 4$, $y = 2(4)-2 = 6$
when $x= -1$, $y = 2(-1)-2 = -4$

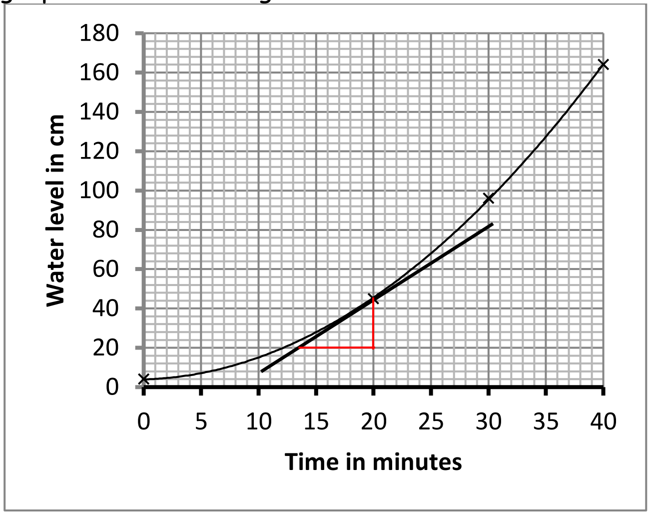
Solutions are:
(4, 6) and (-1, -4)

See points of intersection of graphs for solutions



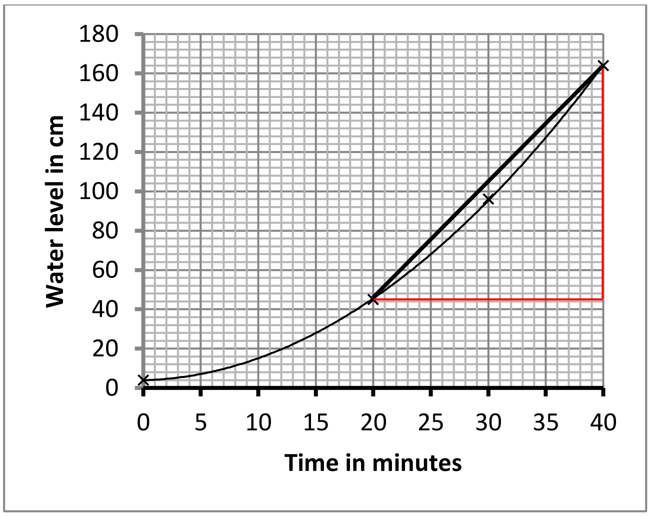
11/20 Interpret gradient of tangent & chord

The instantaneous rate of change of a quantity at a given time is the gradient of the tangent to the graph at that time e.g. 20min



Instantaneous rate of rise of water level at 20min
≈ 24cm ÷ 6.5 min ≈ 3.7cm/min

The average rate of change is the gradient of the chord between the two given times



Average rate of change between 20 & 40min
≈ 120cm ÷ 20min = 6cm/min

11/22 Use iteration to solve equations

Iteration means repeating a process.
Each repetition is called iteration.
The result of an iteration is used as the starting point of the next iteration

e.g. $x^3 - 3x + 1 = 0$

- Write ‘x’ in terms of ‘x’ (here is one way)
  $3x = x^3 + 1$
  $x = \frac{x^3 + 1}{3}$
- Then write as the iteration formula

$x_{n+1} = \frac{x_n^3 + 1}{3}$ (n=previous term; n+1=next term)

- Choose a value for x_1 (it may be given/found from graph
  e.g. $x_1 = 0.2$
- Find x_2 by substituting x_1 into the iteration formula

$x_2 = \frac{(0.2)^3 + 1}{3} = 0.336$

- Find x_3 by substituting x_2 into the iteration formula

$X_3 = \frac{(0.336)^3 + 1}{3} = 0.3459....$

- Continue until answer converges to a given number of d.p. (in this case 0.35(2dp))

Quick method with calculator

- 0.2=
- $(ANS^3+1)\div3=$
- =
- = etc till it converges

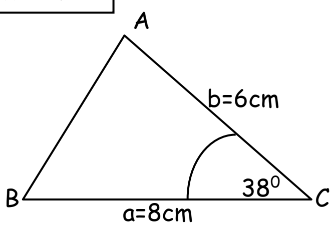
11/22 Area of triangle–height not known

Area = $\frac{1}{2}$ ab sinC
Area = $\frac{1}{2}$ bc sinA
Area = $\frac{1}{2}$ ac sinB

Formula
NOT
provided

Example

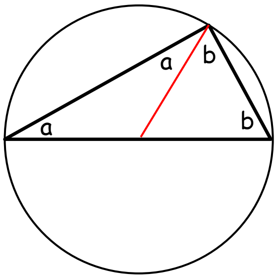
Area = $\frac{1}{2}$ ab sinC
= $\frac{1}{2} \times 8 \times 6 \times \sin 38^0$
= **14.8 cm^2(1dp)**



11/23 Circle Theorem proofs

Angle in a semicircle =90^0

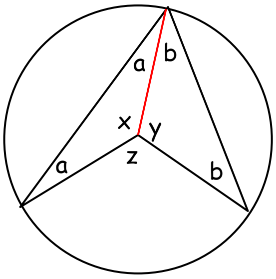
Draw in a radius as shown



In the bold triangle:
2a + 2b = 180 (÷2)
=> a + b = 90^0

Angle at centre = 2x angle at circumference

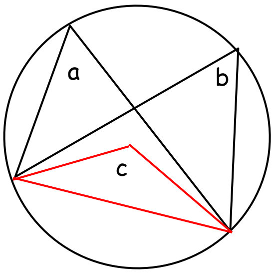
Draw in a radius as shown



x=180 - 2a
y=180 - 2b
z=360 - (x + y)
=360-(360-2a-2b)
=2a + 2b
=2(a + b)

Angles in the same segment are equal

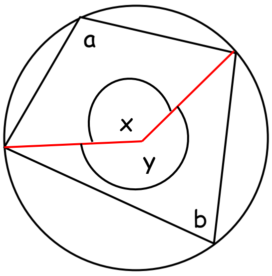
Draw angle at centre from the chord as shown



c=2b (already proved)
c=2a (already proved)
∴ 2a = 2b
∴ a = b

Opposite angles of a cyclic quadrilateral = 180^0

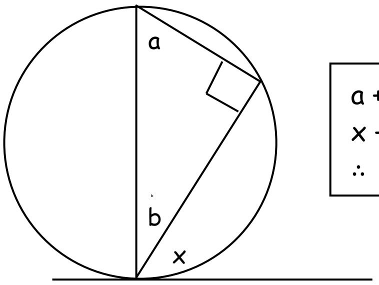
Draw in two radii as shown



x=2b (already proved)
y=2a (already proved)
x + y = 360^0
∴ 2a + 2b = 360^0
∴ a + b = 360^0

Angle between a tangent and its chord is equal to the angle in the 'alternate segment'

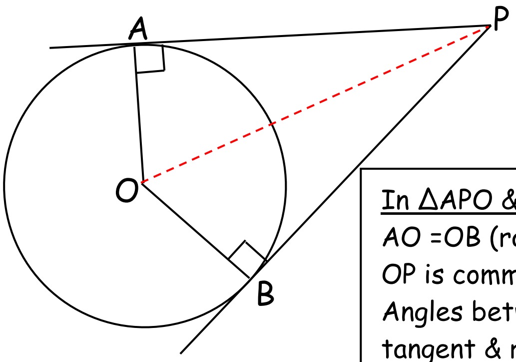
Draw tangent and angle in semicircle as shown



a + b = 90^0
x + b = 90^0
∴ a = x

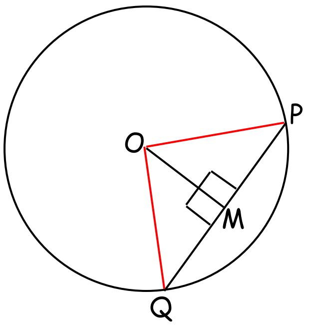
11/23 Circle Theorem proofs (continued)

Equal tangents from a point to the circumference



In ΔAPO & ΔBPO
AO =OB (radii)
OP is common
Angles between tangent & radius = 90^0
ΔAPO & ΔBPO are congruent **(RHS)**
$\therefore \mathbf{AP = BP}$

A radius, perpendicular to a chord bisects the chord

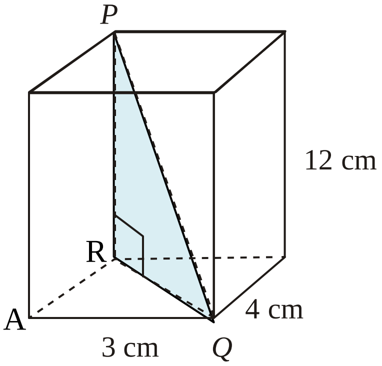


In ΔOPM & ΔAQM
OP =OQ (radii)
OM is common
Angles = 90^0
ΔOPM & ΔAQM are congruent **(RHS)**
$\therefore \mathbf{QM = MP}$

11/24 Use circle theorems

See Stage 10 Prompt Sheet

11/25 Pythagoras Theorem in 3D



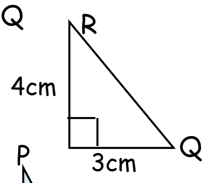
Leave in surd form when needing the EXACT value

Example:

- Identify the triangle in the 3D shape containing the unknown side PQ

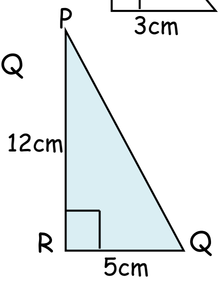
Use Pythagoras in ΔRAQ to find RQ

RQ = $\surd(3^2 + 4^2)$ = 5cm



Use Pythagoras in ΔPRQ to find PQ

PQ= $\surd(12^2 + 5^2)$ = 13cm



$$\text{OR } \sqrt{3^2 + 4^2 + 12^2}$$
$$= \sqrt{169} = 13\text{cm}$$

11/26 Trigonometry in 3D

- Identify the triangle in the 3D shape containing the unknown angle PQR

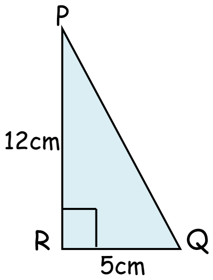
Use Pythagoras in ΔRAQ to find RQ

RQ = $\surd(3^2 + 4^2)$ = 5cm

Use Trigonometry in ΔPRQ to find ∠PQR

Tan PQR = 12 ÷ 5 = 2.4

Tan^{-1}PQR = 67.4^0



11/27 Sine Rule (non-right angled triangles)

Use SINE RULE when given:

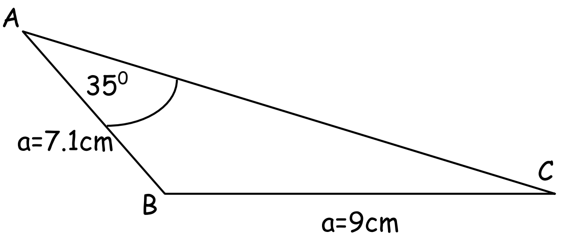
- two sides and a non-included angle
- any two angles and one side

Formula **NOT** provided

To find an angle, use:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example: To find angle C



$$\frac{\sin C}{c} = \frac{\sin A}{a}$$
$$\frac{\sin C}{7.1} = \frac{\sin 35^0}{9}$$
$$\sin C = \frac{\sin 35^0}{9} \times 7.1$$
$$\sin C = 0.4524.....$$
$$C = \sin^{-1}(0.4524.....)$$
$$\underline{C = 28.9^0(1dp)}$$

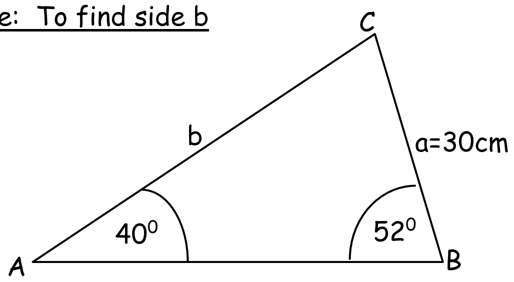
11/27 Sine Rule (continued)

Formula NOT provided

To find a side use:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example: To find side b



$$\frac{b}{\sin B} = \frac{a}{\sin C}$$

$$\frac{b}{\sin 52^0} = \frac{30}{\sin 40^0}$$

$$b = \frac{30}{\sin 40^0} \times \sin 52^0$$

$$\underline{b = 36.8 \text{ cm (1dp)}}$$

11/28 Cosine Rule (non-right angled triangles)

Use COSINE RULE when given:

- 3 sides
- 2 sides and the included angle

Formula NOT provided

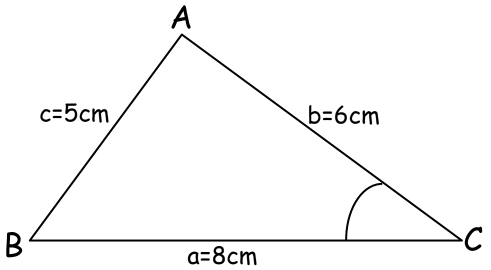
To find an angle, given 3 sides use:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example: To find angle C



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{8^2 + 6^2 - 5^2}{2 \times 8 \times 6}$$

$$\cos C = 0.78125......$$

$$\angle C = \cos^{-1}(0.78125......)$$

$$\angle \underline{C = 38.6^0 (1dp)}$$

To find a side – given 2 sides & included angle use:

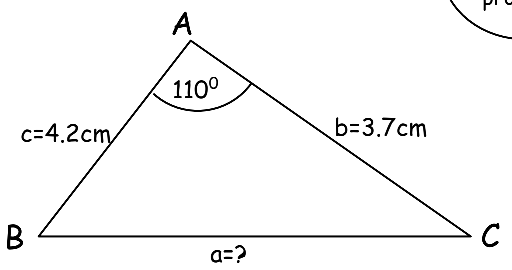
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Formula NOT provided

Example: To find side a



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 3.7^2 + 4.2^2 - 2\times3.7\times4.2 \cos110^0$$

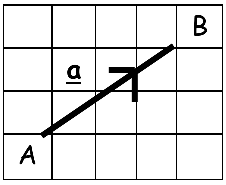
$$a^2 = 41.96$$

$$\underline{a = 6.48(2dp)}$$

11/29 Vectors

- **Vector notation**

This vector can be written a $\binom{3}{2}$ or $\underline{a}$ or $\overrightarrow{AB}$

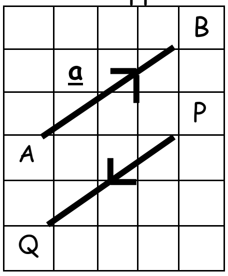


- **A vector has magnitude**(length) & **direction**(shown by an arrow)

Magnitude can be found by Pythagoras Theorem

$$AB = \sqrt{3^2 + 2^2} = \sqrt{13} = 3.6$$

- **A parallel vector** with same magnitude but opposite direction

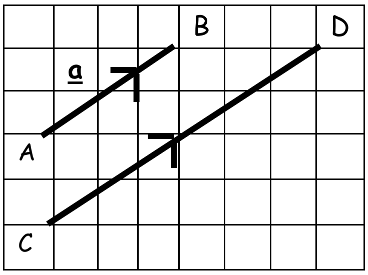


Vector $\overrightarrow{PQ}$ is equal in length to $\overrightarrow{AB}$ but opposite in direction so we say:

$$\overrightarrow{PQ} = -\underline{a}$$

11/29 Vectors (continued)

- **A parallel vector** with same direction but different magnitude



Vector $\overrightarrow{CD}$ is twice (scalar 2) the magnitude but same direction so we say:

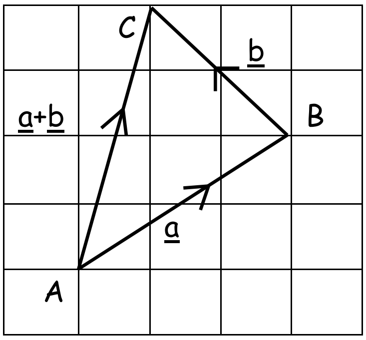
$$\overrightarrow{CD} = 2\underline{a}$$

A negative scalar would reverse the direction

- **Vector addition**

Adding graphically, the vectors go nose to tail

$\underline{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$



The combination of these two vectors:

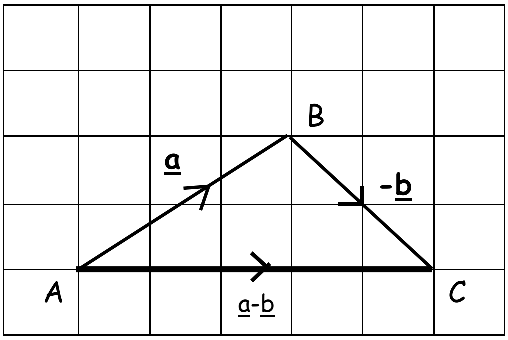
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \underline{\mathbf{a}} + \underline{\mathbf{b}}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

- **Vector subtraction**

$\underline{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$



The combination of these two vectors:

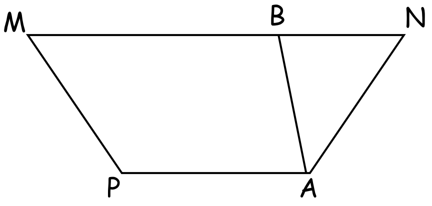
$$\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AC} = \underline{\mathbf{a}} - \underline{\mathbf{b}}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\overrightarrow{AC}$ **is called the RESULTANT vector**

- **The sum of vectors**



$$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{AB}} + \overrightarrow{\mathrm{PM}} + \overrightarrow{\mathrm{MB}}$$

The vector AB is equal to the sum of these vectors or it could be a different route:

go via N

$$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathrm{AN}} + \overrightarrow{\mathbf{NB}}$$

↑ Start point ↑ End point

- **Collinear points (in same straight line)**

To prove 3 points are collinear:

- ✓ Choose two line segments, e.g. AB and BC.
- ✓ Prove that they have:
  - (i) Common direction (equal gradients) and
  - (ii) a common point (e.g. B)

11/30 Histograms

- Class intervals are not equal
- Vertical axis is the **frequency density**
- Frequency is area of bar not the height

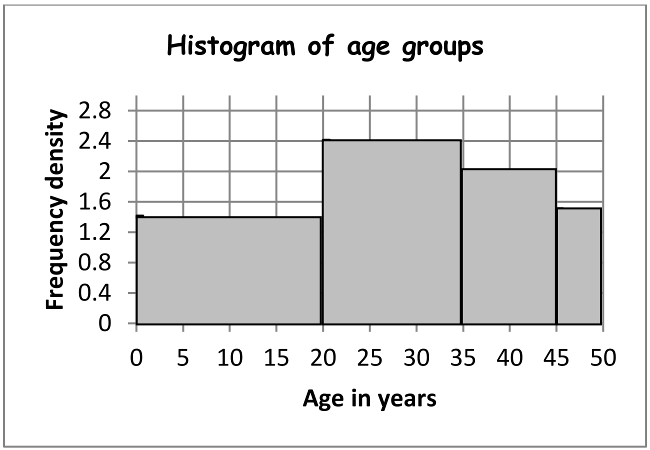
Frequency = class width x frequency density
Frequency density = frequency ÷ class width

- **To draw a histogram**

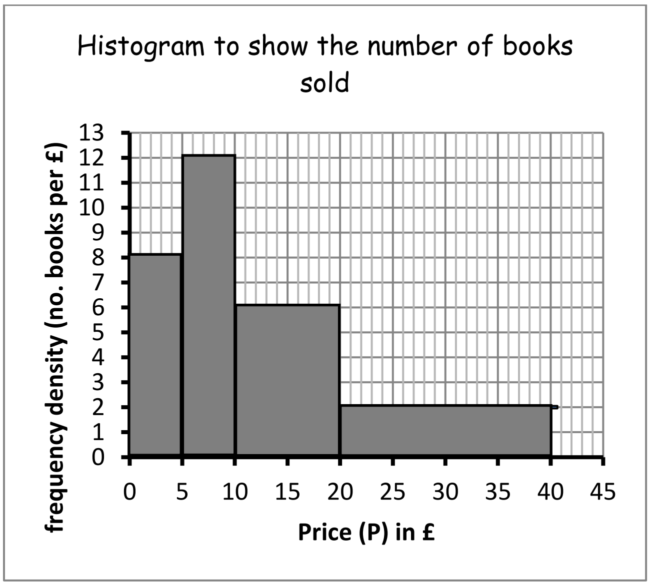
Calculate the frequency density
Example

Age (x years)	Class width	f	Frequency density
$0 < x \le 20$	20	28	28÷20 = 1.4
$20 < x \le 35$	15	36	36÷15 = 2.4
$35 < x \le 45$	10	20	20÷10 = 2
$45 < x \le 65$	20	30	30÷20 = 1.5

Scale the frequency density axis up to 2.4



- **To interpret a histogram**



Price (P) in (£)	f = width x height(fd)
$0 < P \le 5$	5 x 8 = 40
$5 < P \le 10$	5 x 12 = 60
$10 < P \le 20$	10 x 6 = 60
$20 < P \le 40$	20 x 2 = 40